Reminder on higher direct images

$\pi : X \to Y$ morphism of finite type separated k-schemes

$\pi_* : QCoh(X) \to QCoh(Y)$
left exact
$\rightsquigarrow$ δ-functor
$$(R^i\pi_*)_{i \in \mathbb{Z}_{\geq 0}} : QCoh(X) \to QCoh(Y)$$
s.t. $\pi_* = R^0\pi_*$

Explicitly: (if X, Y separated)

(1) $(R^i\pi_*\mathcal{F})(\mathrm{Spec}(B)) := H^i(\pi^{-1}(\mathrm{Spec}(B)), \mathcal{F})$ has structure of B-mod.

$= \cancel{\mathcal{H}^i(\pi_*(\mathcal{F}))}(\mathrm{Spec}(B))$
$\pi_* \mathcal{H}^i(\mathcal{F})$

(for this to define a quasicoherent sheaf need:
$$H^i(\pi^{-1}(\mathrm{Spec}(B)), \mathcal{F})_g = H^i(\pi^{-1}(\mathrm{Spec}(B_g)), \mathcal{F}) \quad)$$
$\forall g \in B$

Properties:

$\pi : X \to Y$ (qc) sep.

a) $\forall$ s.e.s. $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $QCoh(X)$
have functorial l.e.s. in $QCoh(Y)$
$$0 \to R^0\pi_*\mathcal{F}' \to R^0\pi_*\mathcal{F} \to R^0\pi_*\mathcal{F}'' \to R^1\pi_*\mathcal{F}' \to R^1\pi_*\mathcal{F} \to R^1\pi_*\mathcal{F}'' \to \ldots$$

b) if π is proj/proper (e.g. if X, Y proj/proper)
then $R^i\pi_*\mathcal{F} \in Coh(Y)$ for all $\mathcal{F} \in Coh(X)$.

Base change morphism

Consider cartesian diagram of finite type k-schemes and qc sep morphisms.

$$\begin{tikzcd} X' \arrow[r, "\psi'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

$\mathcal{F} \in QCoh(X)$: want a comparison (functorial in $Y' \to Y$)

$$\Phi^i_{Y'} : \psi^* R^i \pi_* \mathcal{F} \longrightarrow R^i \pi'_* \psi'^* \mathcal{F}$$

defined as composition

$$\begin{aligned} \psi^* R^i \pi_* \mathcal{F} &\longrightarrow \pi'_* \pi'^* \psi^* R^i \pi_* \mathcal{F} && (\pi'^*, \pi_*) \text{ adjunction} \\ &= \pi'_* \psi'^* \pi^* R^i \pi_* \mathcal{F} && \text{commutativity of the diagram} \\ &= \pi'_* \psi'^* \pi^* \pi_* \mathcal{H}^i(\mathcal{F}) \\ &\longrightarrow \pi'_* \psi'^* \mathcal{H}^i(\mathcal{F}) && (\pi^*, \pi_*) \text{ adjunction} \\ &\longrightarrow \pi'_* \mathcal{H}^i(\psi'^* \mathcal{F}) && \psi'^* \text{ right exact.} \\ & && \text{if } F \text{ right exact} \\ &= R^i \pi'_* \psi'^* \mathcal{F}. && \exists\, F H^i \longrightarrow H^i F \end{aligned}$$

For $y \in Y$ have cartesian diagram

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ y \arrow[r] & Y \end{tikzcd}$$

and base change morphisms

$$\phi^i_y : (R^i \pi_* \mathcal{F})|_y \longrightarrow H^i(X_y, \mathcal{F}|_{X_y})$$
$$\| $$
$$(R^i \pi_* \mathcal{F})_y \otimes k(y)$$

Examples and counterexamples

- open embeddings are flat.
- torsion sheaves are not flat.

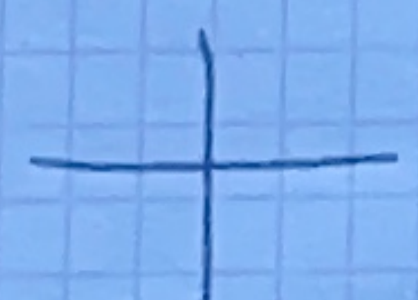
Pf $M \in A\text{-mod}$, torsion. e.g. $x : M \to M$ not injective.

$\Rightarrow M \otimes (0 \to \mathcal{O}_X \xrightarrow{x} \mathcal{O}_X \to \mathcal{O}_X/(x) \to 0)$ not exact. $\square$

- closed embeddings are not flat:

$\mathcal{O}_Z$ $\quad \underset{\text{closed}}{Z \subseteq X} \Rightarrow \mathcal{O}_Z$ ~~not~~ torsion

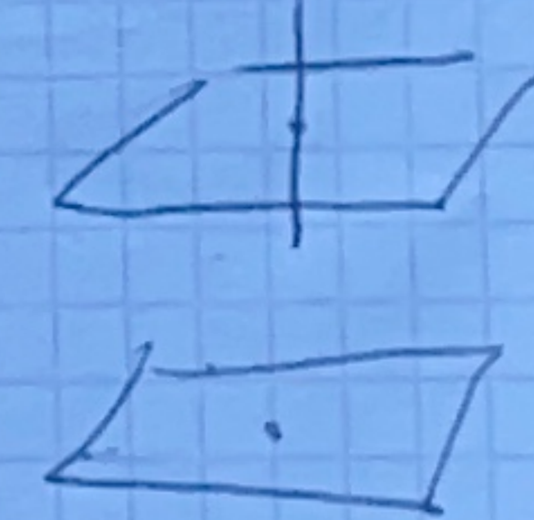
$\Rightarrow Z \hookrightarrow X$ not flat.

-  $\quad \mathrm{Spec}(k[x,y]/(xy)) \longrightarrow \mathrm{Spec}(k[x])$ not flat.

- non-trivial blow ups are not flat

e.g. $Bl_0 \mathbb{A}^2 \xrightarrow{\beta} \mathbb{A}^2$

$$\beta^{-1}(x) = \begin{cases} pt & x \neq (0,0) \\ \mathbb{P}^1 & x = (0,0) \end{cases}$$



- projections are flat.

How to check a morphism / sheaf is flat?

~~Look up criteria in a reference.~~

1) see if numerical invariants are locally constant

2) if yes, look for criteria in references and try to use them to prove flatness

(e.g. "Miracle flatness" at $\underset{\text{Cohen Macaulay}}{X} \longrightarrow \underset{\text{regular}}{Y}$)

Flatness

Def • $\mathcal{F} \in QCoh(X)$ is flat if

$- \otimes_{\mathcal{O}_X} \mathcal{F} : QCoh(X) \to QCoh(X)$

is exact.

~~a~~ $\pi : X \to Y$ morphism

• $\mathcal{F} \in QCoh(X)$ is flat over Y if $\pi_* \mathcal{F} \in QCoh(Y)$ is flat.

• $\pi : X \to Y$ is flat if $\mathcal{O}_X$ is flat over Y

Facts: • compositions of flat morphisms are flat.

• base changes of flat morphisms are flat

• flatness is local on the target.

• tensor product of two flat qcoh sheaves is flat.

Numerical invariants tend to be constant in flat families

Thm if $\pi : X \to Y$ is flat, then

$y \mapsto \dim X_y$

is locally constant on Y

Thm ~~If~~ $\pi : X \to Y$ ~~is a~~ morphism, $\mathcal{F} \in QCoh(X)$ flat over Y,

Then $y \longmapsto \chi(X, \mathcal{F}|_{X_y})$ is locally constant on Y

Cor. $\pi : X \to Y$ ~~flat~~ projective, $\mathcal{L}$ π-rel. ample

$\mathcal{F}$ flat over Y

$\Rightarrow y \longmapsto P_{\mathcal{F}|_{X_y}}(t) \in \mathbb{Q}[t]$ locally constant on Y

<u>Corollary</u>: $\pi : X \to Y$ finite flat morphism
$\mathcal{F} \in Coh(X)$ locally free rank r.
Then 1) $y \mapsto \deg(\pi, y)$ ~~constant~~ locally constant
2) $\pi_* \mathcal{F}$ locally free of rank $\deg(\pi) \cdot r$

<u>Pf</u> π affine $\Rightarrow R^1 \pi_* \mathcal{F} = 0$
$\Rightarrow$ CohBC b) $\Rightarrow \Phi^0_y$ surjective $\forall y \in Y$
CohBC a) $\Rightarrow \Phi^0_y : \pi_* \mathcal{F} \otimes k(y) \xrightarrow{\cong} H^0(X_y, \mathcal{F}|_{X_y})$
CohBC b) $\Rightarrow \pi_* \mathcal{F}$ locally free of rank

$$\underbrace{\dim_{k(y)} (H^0(X_y, \mathcal{O}_{X_y}))}_{= \deg(\pi, y)} \cdot \operatorname{rank}(\mathcal{F})$$

Rank of loc. free sheaf is loc. const $\Rightarrow \deg(\pi, y)$ loc. const. $\square$

Slicing criterion for flatness

$(B, n) \to (A, m)$ morphism of loc. Noeth. alg.
M fin. gen $/A$ $\quad t \in n$ a non-zero divisor.

M is flat over $B \Leftrightarrow$
- $\cdot t : M \to M$ inj.
- M/tM flat over $B/(t)$

<u>Cor</u> $X \to T$ flat, f.t. schemes over k.
$Y \subset X$ a locally principal subscheme.
s.t. $Y_t \subset X_t$ effective Cartier divisor $\forall t \in T$

$\Rightarrow Y \to T$ flat and $Y \subset X$ effective Cartier divisor of X

Behaviour of cohomology in flat families

Semicontinuity theorem

$\pi : X \to Y$ proper, $\mathcal{F} \in \mathrm{Coh}(X)$ flat over Y

Then $\forall i \geq 0$ $Y \to \mathbb{Z}$

$$y \mapsto \dim_{k(y)} H^i(\pi^{-1}(y), \mathcal{F}|_{\pi^{-1}(y)})$$

is upper semicontinuous.

Grauert's theorem

$\pi : X \to Y$ proper, Y reduced.

$\mathcal{F} \in \mathrm{Coh}(X)$ flat over Y

$y \mapsto H^i(\pi^{-1}(y), \mathcal{F}|_{\pi^{-1}(y)})$ locally constant.

$\Rightarrow R^i\pi_* \mathcal{F} \in \mathrm{Coh}(X)$ is locally free and the base change morphism

$\Phi^i_{Y', \mathcal{F}}$ is an isomorphism for all $Y' \to Y$

$\boxed{\text{Cohomology and base change}}$

Cool! Y need not be reduced!

$\pi : X \to Y$ proper, ~~Y loth~~ $\mathcal{F} \in \mathrm{Coh}(X)$ flat over Y

~~Assume~~

Suppose Φ^i_y is surjective for $y \in Y$

Then (1) $\exists$ open nbhd $y \in V \subset Y$ s.t. $\forall \psi : V' \to V$, $\phi^i_{V'}$ is an iso.

(2) TFAE

- Φ^{i-1}_y is surjective
- $R^i\pi_* \mathcal{F}$ is locally free on some open nbhd of y.

Representability of Pic^0 for elliptic curves

Theorem (E,e) elliptic curve over k. $(E, \mathcal{O}_{E\times E}(\Delta) \otimes pr_2^* \mathcal{O}(-e))$ is the Jacobian variety of E.

Proof Need to show:

for all $T \in Sch_{k,ft}$ s.t. for all $t \in T$ $\mathcal{L}|_{E\times\{t\}} \in \mathrm{Pic}^0(E \times \{t\})$ $\exists!$ a morphism $\tau : T \to E$ s.t. $\mathcal{L} \cong \tau^*(\mathcal{O}_{E\times E}(\Delta) \otimes pr_2^*\mathcal{O}(-e))$

$\Updownarrow$

$\mathcal{L}' := \mathcal{L} \otimes p_E^* \mathcal{O}(e) \in \mathrm{Pic}^1_E(T)$ $\quad \mathcal{L}' \cong \tau^* \mathcal{O}_{E\times E}(\Delta)$

$$\begin{tikzcd} & E\times T \arrow[dl, "p_E"'] \arrow[dr, "p_T"] & \\ E & & T \end{tikzcd} \qquad \begin{tikzcd} & E\times E \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ E & & E \end{tikzcd}$$

$\Rightarrow \mathcal{L}'|_{E\times\{t\}}$ degree 1 l.b. on ~~elliptic~~ genus 1 curve

$\Rightarrow h^0(\mathcal{L}'|_{E\times\{t\}}) = 1 \quad h^1(\mathcal{L}'|_{E\times\{t\}}) = 0$

$\mathcal{L}'$ locally free $\Rightarrow \mathcal{L}'$ flat over E
& $\mathcal{L}'$ flat over T

$p_T : E \times T \to T$ is projective
$\Rightarrow$ can apply cohomology and BC.

$H^1(\mathcal{L}'|_{E\times\{t\}}) = 0 \quad \forall t \in T \to \varphi^1_t$ surj $\forall t \in T$;

CohBC $\Rightarrow R^1(p_T)_* \mathcal{L}' = 0$ (loc free)

$\Rightarrow \Phi^0_t$ surjective.

$\Longrightarrow (p_T)_* \mathcal{L}'$ locally free of rank 1.

by ... at Pic... ... about.

replace $\mathcal{L}$ by $\mathcal{L} \otimes p_T^* (p_{T*}(\mathcal{L}'))^\vee$

$\{p_{T*}(\mathcal{L} \otimes p_T^*(p_{T*}\mathcal{L}')^\vee)$
$= \mathcal{O}_T$ proj. formula

$\Rightarrow$ wlog

$(p_T)_* \mathcal{L}' \cong \mathcal{O}_T$

$\Gamma(T, \underbrace{(p_T)_* \mathcal{L}'}_{= \mathcal{O}_T}) \cong \Gamma(\cancel{X} E \times T, \mathcal{L}')$

$1 \longmapsto s$

$Z = Z_s \subset \cancel{X} E \times T$ effective cartier divisor cut out by $s : \mathcal{O}_X \to \mathcal{L}'$ $= \mathcal{L} \otimes \mathcal{O}(e)$

$Z \cap E \times \{t\} \cong q \in E$

$Z \xrightarrow{\bar{p}_T} T$ is flat

(by slicing criterion)

$\deg(Z \cap E \times \{t\}) = 1 \Rightarrow \bar{p}_T$ finite surj. of deg 1.

$\Rightarrow \bar{p}_T$ isomorphism $\overset{\text{cor}}{\Rightarrow} (\bar{p}_T)_* \mathcal{O}_Z = \mathcal{O}_T$

$\Rightarrow \overset{\tau :}{T} \xrightarrow{(\bar{p}_T)^{-1}} Z \to E \times T \xrightarrow{p_E} E$

$Z = \text{graph}(\tau) \Rightarrow \mathcal{L}' \overset{\tau}{=} p_{E} \, \tau^* \mathcal{O}_{E \times E}(\Delta)$

$\square$